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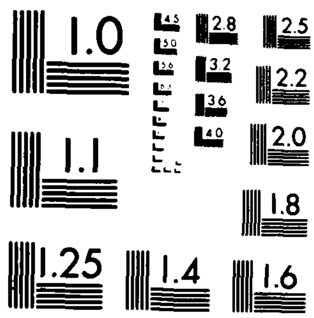

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NOTE ON SUBHARMONIC SOLUTIONS OF A HAMILTONIAN VECTOR FIELD

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ABSTRACT

A forced oscillation problem for a Hamiltonian equation on a torus is studied. If the dimension of the torus is equal to 2n, and if the period of the time dependent Hamiltonian equation is equal to 1, it has been shown in ~~[6]~~, that there are at least (2n+1) periodic solutions having period 1. In this paper it is shown, that, under an additional, necessary nondegeneracy condition such an equation possesses a periodic solution having minimal period T, for every sufficiently large prime number T. The proof uses the classical variational approach. It is based on the Morse theory for periodic solutions developed in [5] which relates the winding number of a periodic solution to its Morse index and on an iteration formula for the winding number.

AMS (MOS) Subject Classifications: 34C25, 58F05,58F22, 58E05, 49H05, 70H05, 70H30

Key Words: Hamiltonian systems, periodic solutions, variational principles, Morse-type index theory, winding number of a periodic solution.

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SIGNIFICANCE AND EXPLANATION

This report is an addition to a previous one concerning periodic solutions of Hamiltonian systems on a torus. In an even earlier work, an analogue of Morse's theorem relating 'conjugate points' on a closed geodesic to the 'index' of the geodesic as a critical point of a functional was proved in the Hamiltonian setting (where the functional is infinitely indefinite and there is no index in Morse's sense).

In this report this analogue of Morse's result is used to find infinitely many periodic solutions on the torus provided they are all linearly non-degenerate. Such theorems all have the aim of understanding the basic action principles of physics.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

NOTE ON SUBHARMONIC SOLUTIONS OF A HAMILTONIAN VECTOR FIELD

C. Conley and E. Zehnder

1. Introduction:

We consider on a torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, $n > 2$, an exact Hamiltonian vectorfield, which depends periodically on time t. Assuming its period to be equal to 1, the vectorfield is, on the universal cover, given by

$$\dot{x} = J\nabla h(t,x), \quad x \in \mathbb{R}^{2n} \quad , \tag{1}$$

where $h \in C^2(\mathbb{R} \times \mathbb{R}^{2n})$ is periodic in all its variables of period 1. One may ask for forced oscillations, i.e. for periodic solutions having the period of the given system. In fact, recently it has been proved in [6] (see also M. Chaperon [3]) that the Hamiltonian vectorfield (1) possesses at least $(2n+1)$ periodic solutions of period 1. This has been conjectured by V. I. Arnold in [1], [2]. Such a periodic solution is also a periodic solution of period $n \in \mathbb{N}$, $n > 1$, $x(t) = x(t+n)$; the period is however not minimal. It is our aim to find solutions having integers $n > 1$ as their minimal periods, such solutions are called subharmonic solutions. To find such solutions further assumptions on the vectorfield are required, as the example $h \equiv 0$ shows.

If $x(t) = x(t+T)$, $T \in \mathbb{N}$ is a T-periodic solution, the linearized equation along $x(t)$ is given by

$$\dot{y} = JS(t)y \quad , \quad S(t) = h_{xx}(t,x(t)) \quad , \tag{2}$$

with $S(t+T) = S(t)$ being symmetric. The fundamental solution $X(t)$ satisfies

$$\dot{X}(t) = JS(t)X(t), \quad X(0) = 1 \quad . \tag{3}$$

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The eigenvalues of $P = X(T)$, T being the period of $x(t)$, are called the Floquet-multipliers of $x(t)$.

Definition: A T-periodic solution $x(t)$ is called non degenerate, if 1 is not an eigenvalue of $P = X(T)$.

This definition requires, that the linearized equation (2) admits no nontrivial T-periodic solution. In the following we shall prove that there are infinitely many subharmonic solutions provided all the periodic solutions with integer periods are nondegenerate.

Theorem 1.

Assume that all the periodic solutions of the Hamiltonian system (1) having integer periods are non-degenerate. Then there is a sequence $\{n_k\}$ of integers with $\lim n_k = \infty$ as $k \to \infty$, such that for every k there are at least 2 periodic solutions having n_k as minimal period.

We point out that the loops of all these periodic solutions are contractible on T^{2n}. Theorem 1 will follow from a more general statement (Theorem 2 below) whose formulation requires some explanation.

2. A Formula for the Iterated Index

First we recall the index of a nondegenerate periodic orbit as introduced in [5]. Let $W = Sp(n,\mathbb{R})$ and let $W^* = \{M \in W \mid 1 \notin \sigma(M)\}$. If $M \in W$ there is a unique polar-decomposition $M = SO$ with S and O belonging to W and S being positive symmetric and O orthogonal. Hence

$$O = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix} \quad \text{with} \quad \bar{u} = u_1 + iu_2 \in U(n) \quad .$$

Here, $U(n)$ is the group of unitary matrices over $\mathbb{C}^n$. Since $|\det \bar{u}| = 1$ there is a homomorphism $O \mapsto \bar{u} \mapsto \det \bar{u} \in S^1$. If $\gamma(t)$, $t_0 \leq t \leq t_1$, is an arc in W there is an associated arc $\bar{u}(t) \in U(n)$. We pick a continuous function

$\Delta(t) \in \mathbb{R}$ with $\det \bar{u}(t) = \exp(i\,\Delta(t))$. Then $\Delta(\gamma) := \Delta(t_1) - \Delta(t_0)$ depends only on γ. If $\gamma(t_0) = \gamma(t_1)$ then $\Delta(\gamma) = 2\pi m$, $m \in \mathbb{Z}$ and the loop γ is contractible in W if and only if $\Delta(\gamma) = 0$.

Consider now a nondegenerate T-periodic solution $x(t)$. Its fundamental solution $X(t) =: \gamma(t)$ according to (3), $0 \leq t \leq T$ is an arc in W satisfying $\gamma(0) = id$ and $\gamma(T) = P \in W^*$. The rotation number of $x(t)$ will be denoted by $\Delta(\gamma)$. Now W^* has two components each of which is simply connected relative to W. One component contains the matrix $Y_+ = -id$ having degree $(+1)$. The other component contains the matrix

$$Y_- = \begin{pmatrix} 2 & & & 0 \\ & -I & & \\ & & \frac{1}{2} & \\ 0 & & & -I \end{pmatrix} \quad ,$$

with I being the identity matrix in $(n-1)$-dimensions. Y_- has degree (-1). We therefore can continue the arc γ by an arc $\hat{\gamma}$ from $\gamma(T) = P \in W^*$ to either Y_+ or Y_-. $\Delta(\hat{\gamma})$ depends only on P and we shall write $\Delta(\hat{\gamma}) = r(P)$. It can be shown, [5], that $0 \leq |r(P)| < \pi n$ and $r(P) = 0$ if

P is hyperbolic. Now $\Delta(\gamma \cup \hat{\gamma}) = \Delta(\gamma) + \Delta(\hat{\gamma})$ is an integer multiple of π and we define the index of the nondegenerate periodic solution $x(t)$ to be the integer

$$j(x) = \frac{1}{\pi}(\Delta(\gamma) + r(P)) \in \mathbb{Z} \quad .$$

If $x(t) = x(t+T)$ is a T-periodic solution we denote the ℓ-times iterated loop $x(t) = x(t+\ell T)$ by x^{ℓ}. If it also is nondegenerate, it has an index $j(x^{\ell})$, namely

$$j(x^{\ell}) = \frac{1}{\pi}(\Delta(\gamma^{\ell}) + r(P^{\ell})) \quad ,$$

where $\gamma^{\ell}(t) = X(t)$, $0 \leq t \leq \ell T$ and $P^{\ell} = X(\ell T) = X(T)^{\ell}$. The following formula then holds true:

<u>Proposition</u>.

<u>Assume the periodic solution</u> x <u>and all its itera</u> <u>s</u> x^{ℓ} <u>are nondegenerate. Then for all</u> $\ell \geq 1$:

(i) $j(x^{\ell}) = \frac{1}{\pi}(\ell\Delta(\gamma) + r(P^{\ell}))$ <u>with</u> $0 \leq |r(P^{\ell})| \leq \pi n$.

(ii) <u>If</u> P <u>is hyperbolic, then</u>:

$$j(x^{\ell}) = \ell \cdot j(x) \quad .$$

As an immediate consequence we have

<u>Corollary</u>. For $\ell \geq 1$

(i) $\left|\frac{\pi}{\ell} j(x^{\ell}) - \Delta(\gamma)\right| \leq \frac{\pi}{\ell} \cdot n$

(ii) <u>Either</u> $\lim_{\ell\to\infty} |j(x^{\ell})| = +\infty$ (<u>in case</u> $\Delta(\gamma) \neq 0$) <u>or</u> $|j(x^{\ell})| \leq n$ <u>for all</u> $\ell \geq 1$ (<u>in case</u> $\Delta(\gamma) = 0$).

<u>Proof</u>.

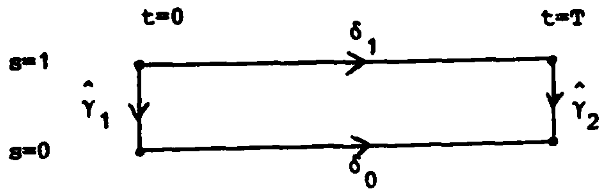
To show $\Delta(\gamma^{\ell}) = \ell\Delta(\gamma)$. Recall that $X(t+\ell T) = X(t)P^{\ell}$, $0 \leq t \leq T$. Now $\Delta(\gamma^{\ell}) = \Delta(\gamma_1) + \Delta(\gamma_2) + \ldots + \Delta(\gamma_{\ell})$, when

$$\gamma_k(t) = X(t)P^{k-1}, \; 0 \leq t \leq T \quad .$$

Consider the polar decompositions $P^{k-1} = e^{A} \cdot O$ and $X(t) = e^{A(t)}O(t)$, and define the deformation δ_s of γ_k by

$$\delta_s(t) = e^{sA(t)}O(t)e^{sA_0},\ 0 < s < 1$$

with $0 < t < T$. From the diagram



where $\delta_1(t) = \gamma_k(t)$ and $\delta_0(t) = O(t)O,\ 0 < t < T$, we conclude that $\Delta(\delta_1 \quad \hat{\gamma}_2 \quad (-\delta_0) \quad (-\hat{\gamma}_1)) = 0$, since the loop is contractible in W. Since $\Delta(\hat{\gamma}_1) = \Delta(\hat{\gamma}_2) = 0$ we conclude $\Delta(\delta_1) = \Delta(\delta_0) = \Delta(O(t)O)$ which is equal to $\Delta(O(t)) = \Delta(\gamma)$. Hence $\Delta(\gamma_k) = \Delta(\gamma)$ and the claim follows.

After these preliminary remarks we can formulate the result of this note.

Theorem 2

If the 1-*periodic solutions of* (1) *are nondegenerate then there are at least* 2^{2n} *of them. Assume that also the iterates of the* 1-*periodic solutions are nondegenerate, then there is an integer* $N_0 > 0$ *such that:*

(i) *for every prime* $T > N_0$ *there is a periodic solution of* (1) *having minimal period* T;

(ii) *moreover, if the* T-*periodic solutions are nondegenerate, then there are at least* 2 *periodic solutions having minimal period* T;

(iii) *if, in addition, for all the* 1-*periodic solutions* $x = x(t)$:

$$\lim_{\ell\to\infty} \frac{1}{\ell}\, j(x^{\ell}) = \Delta(x) \neq 0 \quad ,$$

then there are at least 2^{2n} *periodic solutions having minimal period* T.

All the periodic solutions found are contractible loops on T^{2n}.

The proof is based on the Morse theory for periodic solutions of a time dependent Hamiltonian system as introduced in [5].

3. Morse theory for forced oscillations.

We make use of the classical variational principles for periodic solutions: Let $T > 0$ be an integer, then a T-periodic solution of (1), which is contractible on T^{2n} is a critical point of the functional

$$f(x) = \int_0^T \{\frac{1}{2} \langle \dot{x}, Jx \rangle - h(t,x(t))\}dt \quad ,$$

which is defined on the T-periodic functions $x : [0,T] \to \mathbb{R}^{2n}$. Since $h \in C^2(S^1 \times T^{2n})$, the problem of finding critical points of f can be reduced to the problem of finding critical points of a related function, g, which is defined on the finite dimensional manifold $T^{2n} \times \mathbb{R}^{2N}$ for some large N (which depends on the chosen period T). This reduction is carried out in [6] by means of a global Lyapunov-Schmidt reduction. The required T-periodic solutions are now in one-to-one correspondence to the rest points of the gradient flow

$$\frac{d}{ds} z = \nabla g(z), \; z \in T^{2n} \times \mathbb{R}^{2N} \quad . \tag{4}$$

More precisely this equation can be written as follows, with $z = (y,\xi) \in T^{2n} \times \mathbb{R}^{2N}$.

$$\begin{cases} \frac{d}{ds} y = v_0(z) \\ \frac{d}{ds} \xi = A\xi + v_1(z) \end{cases} ,$$

where v_0 and $v_1 \in C^1(T^{2n} \times \mathbb{R}^{2N})$ are uniformly bounded. Moreover, $A \in (\mathbb{R}^{2N})$ has an invariant splitting $E_+ \oplus E_- = \mathbb{R}^{2N}$ such that for $(\xi_+,\xi_-) \in E_+ \oplus E_-$

$$A\binom{\xi_+}{\xi_-} = \begin{pmatrix} A_+ & \\ & A_- \end{pmatrix}\binom{\xi_+}{\xi_-}$$

with

$$\langle A_+\xi_+,\xi_+ \rangle \geq \frac{2\pi}{T} |\xi_+|^2$$

$$\langle A_-\xi_-,\xi_- \rangle \leq \frac{2\pi}{T} |\xi_-|^2 \quad .$$

It follows that there is a constant $R > 0$ such that

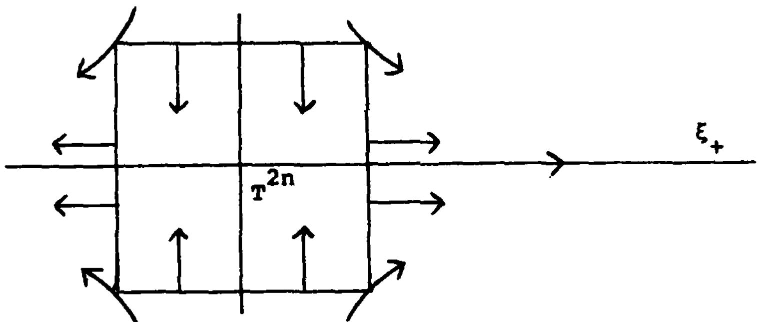
$$\frac{d}{ds} |\xi_+|^2 > 1 \quad \text{if} \quad |\xi_+| > R$$

$$\frac{d}{ds} |\xi_-|^2 < -1 \quad \text{if} \quad |\xi_-| < R \ .$$

Therefore the set S of bounded solutions of (4) is contained in the compact set B,

$$B = T^{2n} \times D_1 \times D_2 \quad ,$$

where $D_1 = \{\xi_+ \in \mathbb{R}^N \mid |\xi_+| < R\}$ and $D_2 \{\xi_- \in \mathbb{R}^N \mid |\xi_-| < R\}$. Moreover $B^- : T^{2n} \times \partial D_1 \times D_2$ is the (strict) exit set of B and $B^+ = T^{2n} \times D_1 \times \partial D_2$ is the (strict) exit set of B for the time reverse flow. We conclude that B is an isolating neighborhood and that the pair (B,B^-) is an index pair for the set S of bounded solutions in the sense of [4].



The index of S, $h(S)$, is the homotopy type $[(B/B^-,*)]$. The algebraic invariants of S are given by the Poincaré-polynomial:

$$\text{(5)} \qquad P(t) := p(t,h(S)) = p(t,B,B^-) = \sum_{j=0}^{2n} \binom{2n}{j} t^{N+j} \quad ,$$

which represents the cohomology of the torus T^{2n}. If there are only finitely many rest points of the flow, z_k, $1 < k < m$, these rest points constitute a Morse-decomposition of the invariant set S. The algebraic invariants $p(t,h(\{z_k\}))$ are related to those of S by the following Morse equation

$$\text{(6)} \qquad \sum_{k=1}^{m} p(t,h(z_k)) = P(t) + (1+t)Q(t) \quad ,$$

with a formal power series Q having nonnegative integer coefficients (e.g., see [5]). If the T-periodic solution $x_k(t)$ which corresponds to the rest point z_k is nondegenerate, then the polynomial $p(t,h(z_k))$ is easily computed:

Lemma ([5], Lemma 2.6)

Assume $x(t) = x(t+T)$ is nondegenerate and denote its index by j. Then the corresponding critical point z of g is an isolated invariant set with index $h(z) = [\dot{S}^m]$, i.e. the homotopy type of a pointed sphere of dimension $m = N+n-j$, so that

$$p(t,h(\{\xi\})) = t^{N+n-j} \quad .$$

Consequently, if all the T-periodic solutions of (1) are nondegenerate ıere are only finitely many of them, $x_1,\ldots,x_m$ having indices $j_1,\ldots,j_m$ - che equation (6) becomes

$$\sum_{k=1}^{m} t^{N+n-j_k} = P(t) + (1+t)Q(t) \quad , \tag{7}$$

with $P(t)$ as in (5). In particular $m \geq 2^{2n}$. The periods may however not be minimal. We shall make use of the proposition in order to estimate the contribution of the iterates of the 1-periodic solutions in equation (7).

4. Proof of theorem 2.

Assume that the 1-periodic solutions of (1) are nondegenerate. There are finitely many of them, $x_1, \dots, x_m$ and we denote their indices by

$$j(x_k) = \frac{1}{\pi}\,(\Delta(x_k) + r(P_k)) \quad .$$

Assume now, in addition, that the iterates of the 1-periodic solutions are nondegenerate. We conclude by the proposition that if $\Delta(x_k) \neq 0$, then the index of the iterated periodic solution increases as $|j(x_k^\ell)| \to +\infty$ as $\ell \to \infty$. On the other hand, if $\Delta(x_s) = 0$ we have $|j(x_s^\ell)| < n$ for all $\ell \geq 1$. Therefore there is an integer $N_0 > 0$ such that for every $\ell \geq N_0$ and for all 1-periodic solutions we have:

$$|j(x_k^\ell)| > n \quad \text{if} \quad \Delta(x_k) \neq 0$$

$$|j(x_s^\ell)| < n \quad \text{if} \quad \Delta(x_s) = 0 \quad .$$

Let $T \geq N_0$ be *prime and assume* there are only finitely many T-periodic solutions, which then constitute a Morse-decomposition of S. By the Lemma we then have the Morse equation

$$\sum_{k=1}^{m} t^{N+n-j_k} + \sum_{s=1}^{r} p(t,h(z_s)) = P(t) + (1+t)\phi(t) \quad ,$$

where the first sum is the contribution of the T-times iterated 1-periodic solutions with indices $j_k = j(x_k^T)$, $1 \leq k \leq m$. Since $|j_k| \neq n$ for every k, then iterated periodic solutions do not represent any cohomology of the torus T^{2n} in dimension zero and in dimension $2n$. Therefore the second term, representing the contribution of the periodic solutions having T as minimal period, is not vanishing. We therefore have at least one critical point, z_s, which corresponds to a periodic solution with minimal period T. If all the T-period solutions are nondegenerate, then by the Lemma we must have $r \geq 2$ in order to represent the lowest and highest cohomology of

T^{2n}. That is there are at least two critical points which correspond to periodic solutions with minimal period T. Finally in case $|j_k| > n$ for $1 < h < m$ we conclude $r > 2^{2n}$ as claimed in the theorem. This finishes the proof of theorem 2.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A forced oscillation problem for a Hamiltonian equation on a torus is studied. If the dimension of the torus is equal to $2n$, and if the period of the time dependent Hamiltonian equation is equal to 1, it has been shown in [6], that there are at least $(2n+1)$ periodic solutions having period 1. In this paper it is shown, that, under an additional, necessary nondegeneracy condition such an equation possesses a periodic solution having minimal period T, for every sufficiently large prime number T. The proof uses the classical variational approach. It is based on the Morse theory for periodic

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ABSTRACT (continued)

solutions developed in [5] which relates the winding number of a periodic solution to its Morse index and on an iteration formula for the winding number.